ISSN: 2454-3659 (P), 2454-3861(E)
Volume I, Issue 7 December 2015
International Journal of Multidisciplinary Research Centre
Research Article / Survey Paper / Case Study

VIZING'S CONJECTURE FOR TOTAL DOMINATION IN GRAPHS

Authors Details
Name: Prof. S.A. Dhanalakshmi
Affiliation: PKR Arts College for women, Gobi
COUNTRY: India

Authors Details
Name: Dr.G.Girija
Affiliation: Government Arts College, Coimbatore
COUNTRY: India

ABSTRACT

In this paper, we study the vizing's conjecture, results related to the conjecture and some open problems related to the conjecture .we give classes of graphs that satisfy this vizing-like inequalities and conjecture for total domination and graph products.

Keywords-Total domination, total domination numbers, Cartesian product, conjecture, cycle, decomposable graph, spanning subgraph, chromatic number, colourable graph.

INTRODUCTION

The results on total domination in Cartesian products of graphs was first introduced by Henning and Rall[4].A set D$\subset V(G)$ is a *total dominating set* if $N(D) = V(G)$. The total domination number is the *minimum cardinality* of a total dominating set of G and is denoted by $\gamma_t(G)$.Henning and Rall conjectured that $2\gamma_t(G\square H) \geq \gamma_t(G)\gamma_t(H)$ and they proved this inequality holds for certain classes of graphs G with no isolated vertices and any graph H without isolated vertices.

Theorem 1.1 [1]

Let G and H be graphs without isolated vertices then $2\gamma_t(G\square H) \geq \gamma_t(G)\gamma_t(H)$

In 1960s, vizing's conjecture was first stated, many results have been published which establish the truth of the conjecture for classes of graphs satisfying certain criteria. Here, we give the classes of graph which are known to satisfy vizing's conjecture.

CLASSES OF GRAPHS SATISFYING VIZING'S CONJECTURE

Vizing's conjecture is that for any two graphs the 2 times total domination number of the Cartesian product of G and H is greater than or equal to the product of the total domination numbers of G and H .The conjecture is stated as follows:

Conjecture 2.1 [2]

For any graphs G and H, $2\gamma_t(G\square H) \geq \gamma_t(G)\gamma_t(H)$. Recall that the Cartesian product of graphs G and H has vertex set

$$V(G\square H) = V(G) \times V(H) = \{(x,y) \mid x \in V(G) \text{ and } y \in V(H)\}$$

and it has edge set

$$E(G\square H) = \{\{(x_1,y_1),(x_2,y_2)\} \mid x_1 = x_2 \text{ and } \{y_1,y_2\} \in E(H);$$
$$\text{or } \{x_1 = x_2\} \in E(G) \text{and} y_1 = y_2 \}.$$

Define a 2-packing of G as a set $X \subset V(G)$of vertices such that $N[x] \cap N[y] = \emptyset$for each pair of distinct vertices $x, y \in X$. alternatively, we can define a 2-packing as a set X of vertices in G such that for any pair of vertices x and y in X, $d(x,y) > 2$. The *maximum cardinality* of a 2-packing of G is called the 2-packing number *of* G and is denoted by $\rho_2(G)$.

Observe that for any graph G, $\rho_2(G) \leq \gamma_t$(G). Let D be a maximal 2-packing of G. Then, as $d(u,v) > 2$for every pair of vertices u and v in D, we need at least one vertex in V(G) to dominate each vertex in S. Hence, the cardinality of a minimal total dominating set is greater than or equal to the cardinality of a maximal 2-packing.

Note that we say a graph G satisfies Vizing's conjecture if, for any graph H, the conjecture inequality holds. Several results establish the truth of Vizing's conjecture for graphs satisfying certain criteria. The case where γ_t(G) = 1 is trivial. A corollary of Barcalkin and German's [9] proof that Vizing's conjecture holds for decomposable graphs is that Vizing's conjecture is true for any graph G with γ_t(G) ≤ 2.

We now consider classes of graphs that are proven to satisfy Vizing's conjecture.

Lemma 2.1[3]

If G satisfies Vizing's conjecture and K *is a spanning* subgraph of G such that $\gamma_t(G) = \gamma_t(K)$, then K satisfies Vizing's conjecture.

Proof

Let K be a spanning subgraph of G obtained by a finite sequence of edge removals which does not change the total domination number. Since K is a subgraph of G, K□H is a subgraph of G□H. Thus we have $2\gamma_t(K\square H) \geq 2\gamma_t(G\square H) \geq \gamma_t(G)\gamma_t(H)$by assumption on G. By assumption on K, we have $\gamma_t(G)\gamma_t(H) = \gamma_t(K)\gamma_t(H)$. We conclude that K satisfies Vizing's conjecture. □

Theorem 2.1 [4]

Let G be a graph and let $x \in V(G)$such that $\gamma_t(G - x) < \gamma_t(G)$. Then if G satisfies Vizing's conjecture, the graph G – x satisfies Vizing's conjecture.

Proof. [4]

Let G be a graph which satisfies Vizing's conjecture, and assume $\gamma_t(G - x) < \gamma_t(G)$for some $x \in V(G)$. Then $\gamma_t(G-x) = \gamma_t(G) - 1$. Now assume there is a graph H such that $2\gamma_t((G-x)\square H < \gamma_t(G - x)\gamma_t(H)$. Let A be a γ_t-set of $(G - x)\square H$ and let B be a γ_t-set of H. Define $S = A \cup \{(x,b) \mid b \in B\}$. Clearly S is a total dominating set of $G\square H$ of cardinality $|A| + |B| < \gamma_t(G-x)(H) + \gamma_t(H) = \gamma_t((G-x)+1)\gamma_t(H) = \gamma_t(G)\gamma_t(H)$. This contradicts our assumption that G satisfies Vizing's conjecture, and so we conclude that $G - x$ satisfies Vizing's conjecture. □

Note that, if the converse of this theorem does not hold, we would have a counterexample to Vizing's conjecture. Consider a graph K that satisfies Vizing's conjecture, and let $S \subseteq V(K)$ be a set of vertices such that no vertex of S belongs to any γ_t-set of K and such that $\gamma_t(K - S) = \gamma_t(K)$. We can form a graph G from K by adding a new vertex v and all edges $\{u,v\}$ where u is in S. If the resulting graph G does not satisfy Vizing's conjecture then obviously we have a counterexample. If, on the other hand, we can prove that the graph G satisfies Vizing's conjecture, then this result would contribute to an attempt to prove Vizing's conjecture by using a finite sequence of constructive operations. The idea is to begin with a class C of graphs for which we know Vizing's conjecture is true and find a collection of operations to apply to graphs from C, each of which results in a graph which satisfies Vizing's conjecture. At this point, the goal would be to show that any graph can be obtained from a seed graph in C by applying a finite set of these operations. This type of approach has obviously not yet been successful, but Hartnell and Rall [3] define several operations which could potentially lead to a proof of Vizing's conjecture using a constructive method.

Lemma 2.2 [5]

For any graphs G and H, $\gamma_t(G\square H) \geq min\{| V(G) |, | V(H) |\}$.

Proof[5]

Let D be a γ_t-set of the product graph G□H, and assume to the contrary that $| D | < min\{| V(G) |, | V(H) |\}$. Then there is a column of vertices Hu= {u}×V(H) and a row of vertices $Gv = V(G) \times \{v\}$ *such that* $D \cap Hu = D \cap Gv = \emptyset$. But then $(u,v) \notin N[D]$, a contradiction. Therefore, $\gamma_t(G\square H) \geq min\{| V(G) |, | V(H |\}$.□

The following result providing a lower bound for $\gamma_t(G\square H)$was proved by Jacobson and Kinch [6].The proof considers a total dominating set for the product graph $G\square H$ and counts the way that the total dominating set intersects each set of vertices $V(G) \times \{v\}$, *where* $v \in V(H)$.

Theorem 2.2[6]

For any graphs G and H , $\gamma_t(G\square H) \geq \frac{|H|}{\Delta(H)+1}\gamma_t(G)$.

Observe that this theorem implies Vizing's conjecture holds for cycles of length 3k. Consider the Cycle

C_{3k}, for $k \geq 1$ an integer. We have $\Delta(C_{3k}) = 2$ and $\gamma_t(C_{3k}) = k$, so therefore $\frac{|C_{3k}|}{\Delta(C_{3k})+1} = \frac{3k}{3} = k = \gamma_t(C_{3k})$.

Theorem 2.3[7]

For any graphs G and H, $\gamma_t(G\square H) \leq min\{\gamma_t(G) | V(H) |, | V(G) | \gamma_t(H)\}$.

Proof.

Let A be a γ_t-set of G. Now let $D = \{A \times \{v\} | v \in V(H)\}$. then D is a total dominating set of $G\square H$ of cardinality $\gamma_t(G) | V(H) |$. Similarly, we can let B be a γ_t-set of H and define $D = \{\{u\} \times B \mid u \in V(G)\}$. Thus, we have $\gamma_t(G\square H) \leq min\{\gamma_t(G) | V(H) |, | V(G) | \gamma_t(H)\}$. □

Theorem 2.4[8]

For any graphs G and H,

$$\gamma_t(G\square H) \geq max\{\gamma_t(G)\_2(H), \_2(G)\gamma_t(H)\}.$$

Theorem 2.5 [9]

For any graphs G and H,

$$\gamma_t(G\square H) \geq \gamma_t(G)\rho_2(H) + \rho_2(G)\gamma_t((H) - \rho_2(H)).$$

The earliest significant result related to the domination number of a Cartesian product was produced by Barcalkin and German [4] in 1979. Barcalkin and German studied graphs G which have domination number equal to the chromatic number of $\bar{G}$. Recall that the chromatic number $\chi(G)$of a graph G is the smallest number of colors needed to color the vertices of G in such a way that no two adjacent vertices are the same color. Observe that any proper coloring of $\bar{G}$ is a partition of the vertices of G into cliques, or complete subgraphs of G. A single vertex may be chosen from each clique to form a total dominating set of G and, therefore, it is always true that $\gamma_t(G) \leq \chi(\bar{G})$.

Barcalkin and German defined *decomposable graphs* as follows. Let G be a graph with $\gamma_t(G) = k$, and assume $V(G)$can be partitioned into k sets $C_1, C_2, \ldots, C_k$ such that each induced subgraph $G[C_i]$is a complete subgraph of G. If G satisfies these conditions, then it is a decomposable graph. They also define the *A-class*, which consists of all graphs G'that are spanning subgraphs of a decomposable graph G, where $\gamma_t(G') = \gamma_t(G)$. The result of Barcalkin and German's 1979 paper established Vizing's conjecture for any graph which belongs to the A-class. Note that we now commonly refer to this class of graphs as BG-graphs.

Theorem 2.6 [10]

Let G be a decomposable graph and let K be a spanning subgraph of G with $\gamma_t(G) = \gamma_t(K)$. Then K satisfies Vizing's conjecture.

Proof. [9]

We assume that G is a decomposable graph with $\gamma_t(G) = k$. Let $\{C_i \mid G[C_i]$ is a complete subgraph of $G, 1 \le i \le k\}$ be a partition of V(G).We now consider the partition $\{C_i \times V(H) \mid i = 1, \dots, k\}$ of $V(GH)$ for H an arbitrary graph. Let D be a γ_t-set of $G\square H$.

Denote by D_j the set of vertices in D that are also in $C_j \times V(H)$. That is,

$$D_j = D \cap \left(C_j \times V(H)\right) for\ j = 1, \dots, k.$$

Let $u_j \in C_i$ and denote by P_i the projection of vertices in $C_i \times V(H)$onto $\{u_j\} \times V(H)$. Let L_j be the set of all vertices v such that $(u_j, v)$is not total dominated by $P_j(D_j)$. That is,

$L_j = \{v \mid (u_i, v) \notin N[P_j(C_j)]\}$.

We observe that if $v \in L_i$, then the vertices $C_i \times \{v\}$ are total dominated "horizontally". Obviously, if $P_j(D_j)$ totally dominates $u_j \times V(H), \mid L_j \mid$=0. However, if $\mid D_j \mid = \gamma(H) - m$ then we have

$$\mid D_j \mid + \mid L_j \mid \ge \mid P_j (D_j) \mid + \mid L_j \mid \ge \gamma_t(H).$$

This implies that $|L_j| \ge$ m.

We now consider $v \in V(H)$ such that $v \in L_j$ for at least one i =1, . . . , k. Define the sets D_v, S_v and $A_v$as follows. We let $S_v = \{\mid v \in L_i and\ i = 1, \dots, k\}$. Define A_v to be the set of cliques C_j such that there is at least one edge from a vertex in C_j to a member of S_v and $D \cap (C_j \times \{v\}) \neq \emptyset$. Finally, we let $D_v = \{u \in V(G) \mid (u, v) \in D\ and\ u \in C_j \in Av\}$.

We observe that $\mid D_v \mid \ge \mid S_v \mid + \mid A_v \mid$, for otherwise we would have

$$D\hat{}v = D_v \cup \{(C_j, v) \mid C_j \notin S_v \cup A_v\}$$

is a dominating set of $V(G) \times \{v\}$of cardinality less than k. Also observe that for each $i = 1, \dots, k$ either $\mid Di \mid \ge (H)$, in which case summing over I gives the desired inequality; or $\mid D_j \mid = (H) - m$. in the latter case, we have shown that $\mid D_v \mid \ge \mid S_v \mid + \mid A_v \mid$. from this, we have

$$\mid S_v \mid \le X \sum_{u \in Dv} (\mid D \cap (C_j \times \{u\}) \mid - 1). \quad (2.1)$$

Thus, we have sufficient extra vertices in D in neighboring cliques so that we still have an average of γ_t (H) for each $|D_j|$.

We conclude that

$$|D| \ge \gamma_t(G)\, \gamma_t(H)$$
$$2|D| \ge |D|$$
$$2|D| \ge \gamma_t(G)\, \gamma_t(H)$$
$$2\gamma_t(GH) = 2|D| \ge \gamma_t(G)\, \gamma_t(H)$$

Corollary 2.1[9]

Let G be a graph satisfying γ_t(G) = 2 *or* ρ_2(G) = γ_t (G). Then G satisfies Vizing's conjecture.

This corollary follows from the previous theorem. Any graph G with

$\gamma_t(G) = 2$ is a subgraph of a decomposable graph. To establish the second part of the corollary, we assume G is a graph satisfying γ_t (G) = ρ_2 (G).

Let $S = \{v_1, v_2, \ldots, v_k\}$be a 2-packing of G. Then we can add edges to G to make $N[v_1], N[v_2], \ldots, N[v_k - 1]$ and $V(G) - (N[v_1] \cup N[v_2] \cup \ldots \cup N[v_k - 1])$ into cliques.The resulting graph is decomposable and still has k pairwise disjoint closed neighbourhoods. Hence,it follows from Theorem 2.6 that any graph with $\gamma_t(G) = \rho_2(G)$satisfies Vizing's conjecture. An example of this can be seen in Figure 1. The labelled verticesv_1,v_2, and v_3 in G form a 2-packing of the graph. We can add edges as described above to get the decomposable graph H.

A graph G with $\gamma_t(G) = \rho_2(G)$and a decomposable graph H formed by adding edges to G.

Observe that this corollary implies Vizing's conjecture is true for any tree.

Corollary 2.2 [4]

Let G be a graph such that $\bar{G}$*is* 3-colorable. *Then* G satisfies Vizing's conjecture.

Proof.

We consider three cases based on the chromatic number of $\bar{G}$

- Case 1: $\chi(\bar{G})$= 1. Then G is a complete graph and the result holds.
- Case 2: $\chi(\bar{G})$ = 2. Then G belongs to the A-class and Vizing's conjecture holds.
- Case 3: $\chi(\bar{G})$=3. If $\gamma_t\ (G) = 3$ then G is decomposable and result holds by Theorem 2.6.Otherwise $\gamma_t(G) \leq 2$ and result holds by Corollary 2.1.

We now define Type χ graphs, as introduced by Hartnell and Rall in 1995. This class of graphs contains the BG-graphs as a proper subset and, hence, is an improvement of Barcalkin and German's 1979 result. Hartnell and Rall, in defining Type χ graphs, took an approach similar to that of Barcalkin and German in that they considered a particular way of partitioning a graph G.

The difference is that not every set in the partition of a Type χ graph induces a complete subgraph .Type χ graphs are defined as follows. Let k, t, r be nonnegative integers, not all zero. Let G be graph with $\gamma_t(G) = k + t + r + 1$ whose vertices can be partitioned as $S \cup SC \cup BC \cup C$, where

S, SC, BC, and C satisfy the following.

- Let $BC = B1 \cup B2 \cup \ldots \cup$Bt. Each Bi for $i = 1, \ldots, t$ is referred to as a *buffer clique*.
- Let $C = C_1 \cup C_2 \cup \ldots \cup C_r$.
- Each of $SC, B_1, \ldots, B_k, C_1, \ldots, C_r$ induces a clique.
- Every $v \in SC$ has at least one neighbour outside of SC. The set SC is called a *special clique*.
- Each B_i, for $i = 1 \ldots, k$ has at least one vertex which has no neighborus outside of Bi.
- Let $S = S_1 \cup S_2 \cup \ldots \cup S_k$ where each S_i is star-like. That is, each S_i has a vertex $v_i$which is adjacent to all $v \in S_i - V_i$. The vertex V_i has no neighbours other than those in.S_i Note that S_i does not induce a clique, and no edges may be added to Si without decreasing the total domination number of G.
- There are no edges between vertices in S and vertices in C.

Observe that not every graph that is Type χ has a special clique. We can also have t, r, or k equal to zero. The example in Figure 2, is a Type χ graph with a special clique. In this graph, the blue vertices represent the set S, the red vertices represent the buffer clique B, and the green vertices represent the special clique SC. One can easily verify that this graph satisfies the definition of Type χ graphs above.

Theorem 2.7 [10]

LetG be a Type χ graph. Then for any graph H, $\gamma_t(G\Box H) \geq \gamma_t(G)\,\gamma_t(H)$.

The proof of Hartnell and Rall's theorem is similar to the proof that Vizing's conjecture is truefor BG-graphs. We partition the vertices of G as indicated by the definition of a Type χ graph and consider any total dominating set D of $G\Box H$. Hartnell and Rall used the idea that some vertices in the product graph must be dominated "horizontally" and found (G) disjoint sets in D, each of which have cardinality at least (H), thus implying that Vizing's conjecture holds for any Type χ graph. Similarly, By this idea, that some vertices in the product graph must be total dominated "horizontally" and found (G) disjoint sets in D, each of which have cardinality at least (H), thus implying that Vizing's conjecture holds for any Type χ graph.

Theorem 2.8[10]

Let G be a Type χ graph and let K be a spanning subgraph of G such that $\gamma_t(G) = \gamma_t(K)$. Then Vizing's conjecture is true for K.

This theorem can be proved in the same way we showed that any spanning subgraph K of a decomposable graph G with $\gamma_t(G) = \gamma_t(K)$ satisfies Vizing's conjecture.

Hartnell and Rall were also able to show that any graph with domination number one more than its 2-packing number is a Type x graph and, hence, we have the following result for total domination also .

Corollary 2.3 [10]

Let G be a graph satisfying $\gamma_t(G) = \rho_2(G) + 1$.Then Vizing's conjecture is true for G.

CONCLUSION

The classes of graphs which are known to satisfy vizing's conjecture were described and the vizing-like conjecture for total domination have also been studied.

REFERENCES

1.W. Edwin Clark and Stephen Suen, An Inequality Related to Vizing's Conjecture. *The ElectronicJournal of Combinatorics*, 2000

2.V. G. Vizing, Some unsolved problems in graph theory. *Uspehi Mat. Nauk*23:117-134, 1968.

3.B. L. Hartnell and D. F. Rall, On Vizing's conjecture. *Congr. Numer.* 82 87-96, 1991

4.Bert Hartnell and Douglas F. Rall, Domination in Cartesian Products: Vizing's Conjecture. *Domination in Graphs*. Edited by TeresaW. Haynes, Stephen T. Hedetniemi and Peter J. Slater.163-189, 1998.

5.M. El-Zahar and C. M. Pareek, Domination number of products of graphs. *ArsCombin.* 31 223-227, 1991.

6.M. S. Jacobson and L. F. Kinch, On the domination number of products ofgraphs: I. *ArsCombin.* 18: 33-44, 1983.

7. V. G. Vizing, The Cartesian product of graphs. *Vy˘cisl. Sistemy*9: 30-43, 1963

8.G. Chen, W. Piotrowski and W. Shreve, A partition approach to Vizing's conjecture. *J. Graph Theory* 21 103-111, 1996.

9.M. Barcalkin and L. F. German, The external stability number of the Cartesian product of graphs. *Bul. Akad. ˘Stiince RSS Moldoven*. no.1 5-8, 1979.

10. B. L. Hartnell and D. F. Rall, Vizings conjecture and the one-half argument. *Discuss. Math. Graph Theory*, 15(2):205-216, 1995.

FIGURES

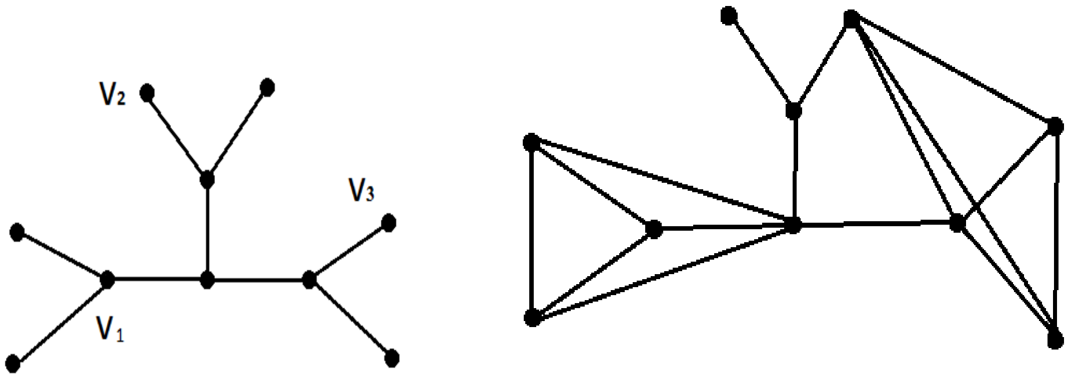

Figure1

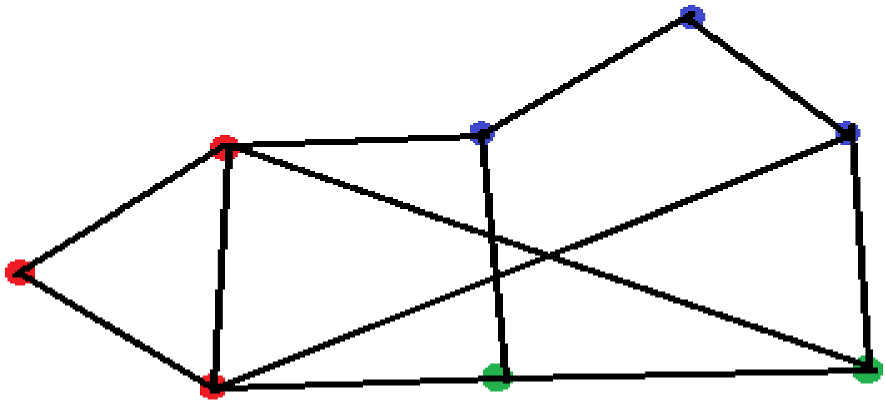

Figure 2.: Example of a Type χ graph with a special clique